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COMMUTATIVE TORSION THEORY

BY

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ABSTRACT. This paper links several notions of torsion theory with commutative concepts. The notion of dominant dimension [H. H. Storrer, *Torsion theories and dominant dimensions*, Appendix to Lecture Notes in Math., vol. 177, Springer-Verlag, Berlin and New York, 1971. MR 44 #1685.] is shown to be very close to the notion of depth. For a commutative ring A and a torsion theory such that the primes of A, whose residue field is torsion-free, form an open set U of the spectrum of A, Spec A, a concrete interpretation of the module of quotients is given: if M is an A-module, its module of quotients $Q(M)$ is isomorphic to the module of sections $\widetilde{M}(U)$, of the quasi-coherent module $\widetilde{M}$ canonically associated to M. In the last part it is proved that the (T)-condition of Goldman is satisfied [O. Goldman, *Rings and modules of quotients*, J. Algebra 13 (1969), 10–47. MR 39 #6914.] if and only if the set of primes, whose residue field is torsion-free, is an affine subset of Spec A, together with an extra condition. The extra, more technical, condition is always satisfied over a Noetherian ring, in this case also it is classical that the (T)-condition of Goldman means that the localization functor Q is exact. This gives a new proof to Serre's theorem [J.-P. Serre, *Sur la cohomologie des variétés algébriques*, J. Math. Pures Appl. (9) 36 (1957), 1–16. MR 18, 765.]. As an application, the affine open sets of a regular Noetherian ring are also characterized.

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0. Well-centered torsion theory. All the rings are supposed commutative and unitary. We denote by $(\mathcal{T}, \mathcal{F})$ a torsion theory over a ring A. We recall that if $\mathfrak{p}$ is a prime of A, $A/\mathfrak{p}$ is either a torsion module, and we say that $\mathfrak{p}$ is a torsion-prime, or a torsion-free module, and we say that $\mathfrak{p}$ is a free-prime. Therefore the spectrum Spec A is partitioned into two subsets $\mathbf{T}$ and $\mathbf{F}$; the set $\mathbf{T}$, set of torsion-primes is closed under specialization (and its complement $\mathbf{F}$ is, of course, closed under generization). We denote by $\mathrm{Ass}_A(M)$ the set of primes weakly associated to an A-module M [1, Chapter IV, Exercise 17]; if $\mathrm{Ass}_A(M) \subset \mathbf{F}$ then M is torsion-free, whereas if M is torsion $\mathrm{Ass}_A(M) \subset \mathbf{T}$. The converses of these two statements do not hold in general. When they do hold we say that $(\mathcal{T}, \mathcal{F})$ is well centered. In fact, for $(\mathcal{T}, \mathcal{F})$ to be well centered it is enough that $\mathrm{Ass}_A(M)$

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$\subset \mathbf{F}$, for every torsion-free module M. If the ring A is Noetherian, every torsion theory is well-centered; in this case there is a bijection between the set of torsion theories over A, and the subsets $\mathbf{F}$ of Spec A, closed under generization. (For all these preliminaries cf. [2].)

A. DEPTH

1. **Dominant dimension.** Let A be a ring with a torsion theory $(\mathcal{T}, \mathcal{F})$, following Tachikawa [18] and Storrer [16] we introduce the definition:

Definition 1.1. A module M is said to have $(\mathcal{T}, \mathcal{F})$-dominant dimension $\geq n$, if there exists an injective resolution of M whose first n-terms are torsion-free.

We denote the $(\mathcal{T}, \mathcal{F})$-dominant dimension of M by $(\mathcal{T}, \mathcal{F})\text{-}d_A(M)$; it is an integer or the symbol ∞.

The following results are immediate [16]:

1.2. $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) \geq n$ if and only if the first n terms of the minimal injective resolution of M are torsion-free.

1.3. $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) \geq 1$ if and only if M is torsion-free.

1.4. $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) \geq 2$ if and only if M is torsion-free and divisible.

We give now another approach to dominant dimension. To every module corresponds its torsion radical $T(M)$. Clearly, if $f: M \to N$ is a module morphism, the restriction $T(f)$ of f to $T(M)$ takes $T(M)$ into $T(N)$, hence T can be regarded as a functor. It is easy to show that T is left exact as a functor from the category of A-modules to itself. If $0 \to M_0 \to M_1 \to \cdots \to M_n \to$ is an injective resolution of M, it is classical to define the nth derived functor T_n of T as the nth homology group of the complex $0 \to T(M_0) \to T(M_1) \to \cdots \to T(M_n) \to \cdots$ and then T_0 is functorially isomorphic to T [14].

Proposition 1.5. $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) \geq n+1$ *if and only if* $T_k(M) = 0$, $\forall k \leq n$.

Proof. From 1.3. $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = 0$ if and only if M is not torsion-free, that is $T(M) = T_0(M) \neq 0$. Conversely if M is torsion-free, we denote the injective hull of M by $I(M)$ and the quotient $I(M)/M$ by M'; we consider the exact sequence $0 \to M \to I(M) \to M' \to 0$, it gives rise to a long exact sequence

$$0 \to T_0(M) \to T_0(I(M)) \to T_0(M') \to T_1(M) \to \cdots$$

$$\to T_k(M) \to T_k(I(M)) \to T_k(M') \to T_{k+1}(M) \to \cdots$$

and clearly $T_k(M) = 0\ \forall k \leq n$ if and only if $T_k(M') = 0\ \forall k \leq n-1$; it is also clear from the definition that $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = (\mathcal{T}, \mathcal{F})\text{-}d_A(M') + 1$. Hence, the proposition would follow from an easy induction on the dominant dimension of M, going from M' to M.

The following proposition is trivial but worth noting:

Proposition 1.6. *If* $(\mathcal{T}', \mathcal{F}')$ *is a torsion theory smaller than* $(\mathcal{T}, \mathcal{F})$ (*that is to say that every torsion-free module for* $(\mathcal{T}, \mathcal{F})$ *is also torsion-free for* $(\mathcal{T}', \mathcal{F}')$), *then, for every module* M, $(\mathcal{T}', \mathcal{F}')\text{-}d_A(M) \geq (\mathcal{T}, \mathcal{F})\text{-}d_A(M)$.

2. **Depth.** We let now A be a Noetherian ring. If M is an A-module, and I an ideal of A, a sequence $f_1, f_2, \cdots, f_n$ of elements of I is called an M-regular sequence in I if f_1 is not a zero divisor in M, and f_{i+1} is not a zero divisor in $M/f_1M + \cdots + f_iM$, $1 \leq i \leq n-1$. The I-Depth of M, denoted by $I\text{-Depth}_A(M)$ is the maximal length of an M-regular sequence in I; if A is a local ring and if $\mathfrak{m}$ denotes its maximal ideal; one writes $\text{Depth}_A(M)$ for $\mathfrak{m}\text{-Depth}_A(M)$ and calls it simply the depth of M [11, Chapter 6]. On the other hand there exists a torsion theory $(\mathcal{T}_I, \mathcal{F}_I)$ corresponding to the partition $\mathbf{T}$, $\mathbf{F}$, where $\mathbf{T} = V(I) = \{\mathfrak{p} \in \text{Spec } A \mid I \subset \mathfrak{p}\}$, since $V(I)$ is closed under specialization [§ 0]. Here is the main theorem of this section.

Theorem 2.1. *Let* A *be a Noetherian ring and* I *be an ideal of* A. *For every* A*-module* M *of finite type*: $(\mathcal{T}_I, \mathcal{F}_I)\text{-}d_A(M) = I\text{-Depth}_A(M)$.

Proof. The proof is in every respect similar to Matsumara's proof [11, 15.B, Theorem 26], that $\text{Ext}^i_A(A/I, M) = 0$ for any $i < n$ if and only if there exists an M-regular sequence $f_1, \cdots, f_n$ in I. Show by induction that there exists such an M-regular sequence if and only if $T_i(M) = 0$, for any $i < n$ (where T is the torsion radical functor relative to $(\mathcal{T}_I, \mathcal{F}_I)$, $T_1, T_2, \cdots, T_n, \cdots$ its derived functors). We show only the first step:

Suppose that $T_0(M) = T(M) = 0$, hence that M is torsion-free for $(\mathcal{T}_I, \mathcal{F}_I)$. Then $\text{Ass}_A(M)$ does not meet $\mathbf{T} = V(I)$. No prime associated to M contains I. So I is not included in the finite union of these primes and there is an element f_1 in I which is not a zero divisor in M.

Conversely suppose that there is such an element f_1 in I. f_1 cannot be contained in any prime associated to M, $\text{Ass}_A(M)$ must be included in $\mathbf{F}$, M must be torsion-free.

3. **Change of rings.** Let A be a ring with a torsion theory $(\mathcal{T}, \mathcal{F})$ and ϕ: $A \to B$ a ring morphism. The direct image of $(\mathcal{T}, \mathcal{F})$ by ϕ is a torsion theory $(\mathcal{T}^\phi, \mathcal{F}^\phi)$ for B-modules. It is the theory such that a B-module is regarded as a torsion module if and only if its inverse image $\phi_*(M)$ is a torsion module for the theory $(\mathcal{T}, \mathcal{F})$ over A. It is easy to check that the B-torsion free modules are the modules M such that $\phi_*(M)$ is torsion-free for $(\mathcal{T}, \mathcal{F})$. We state the two following results without proof:

An ideal $\mathfrak{b}$ of B is in the idempotent filter associated to $(\mathcal{T}^\phi, \mathcal{F}^\phi)$ [9, Proposition 0.4] if and only if $\phi^{-1}(\mathfrak{b})$ is in the idempotent filter associated to $(\mathcal{T}, \mathcal{F})$.

If $(\mathbf{T}, \mathbf{F})$ is the partition of Spec A corresponding to $(\mathcal{T}, \mathcal{F})$, and $(\mathbf{T}^\phi, \mathbf{F}^\phi)$ the partition of Spec B corresponding to $(\mathcal{T}^\phi, \mathcal{F}^\phi)$, then $q \in \mathbf{T}^\phi \Leftrightarrow \phi^{-1}(q) \in \mathbf{T}$, and $q \in \mathbf{F}^\phi \Leftrightarrow \phi^{-1}(q) \in \mathbf{F}$.

Proposition 3.1. *Let A be a ring with a torsion theory $(\mathcal{T}, \mathcal{F})$, T the corresponding torsion radical functor and $T_1, T_2, \cdots, T_n, \cdots$ its derived functors. Let $\phi: A \to B$ be a ring morphism such that B is a flat A-module, and T^ϕ the torsion radical functor corresponding to $(\mathcal{T}^\phi, \mathcal{F}^\phi)$, $T_1^\phi, T_2^\phi, \cdots, T_n^\phi \cdots$ its derived functor. Let M be a B-module and $\phi_*(M)$ its inverse image; then*

$$\phi_*(T_n^\phi(M)) \cong T_n(\phi_*(M)) \quad \text{for all } n$$

and in particular $(\mathcal{T}, \mathcal{F})\text{-}d_A(\phi_*(M)) = (\mathcal{T}^\phi, \mathcal{F}^\phi)\text{-}d_B(M)$.

Proof. Very easily $\phi_*(T^\phi(M)) \cong T(\phi_*(M))$, and since the inverse image of an injective resolution of M is an injective resolution of $\phi_*(M)$ (since ϕ is flat) the isomorphism holds for all n.

If S is a multiplicative set of a Noetherian ring A, and I is an injective A-module, $S^{-1}I$ is an injective $S^{-1}A$-module [16, Proposition 7.17 and Corollary 7.14], then

Proposition 3.2. *Let A be a Noetherian ring with a torsion theory $(\mathcal{T}, \mathcal{F})$, T the corresponding torsion radical functor and $T_1, T_2, \cdots, T_n, \cdots$ its derived functors. Let S be a multiplicative set of A and $(S^{-1}\mathcal{T}, S^{-1}\mathcal{F})$ be the direct image of $(\mathcal{T}, \mathcal{F})$ under the canonical morphism $A \to S^{-1}A$. Let $(S^{-1}T)$ be the torsion radical functor associated to this theory, $(S^{-1}T_1), (S^{-1}T_2), \cdots, (S^{-1}T_n), \cdots$ its derived functors. Let M be an A-module; then*

$$S^{-1}[T_n(M)] \cong T_n(S^{-1}M) \cong (S^{-1}T_n)(S^{-1}M) \quad \text{for all } n$$

and in particular

$$(\mathcal{T}, \mathcal{F})\text{-}d_A(M) \leq (\mathcal{T}, \mathcal{F})\text{-}d_A(S^{-1}M) = (S^{-1}\mathcal{T}, S^{-1}\mathcal{F})\text{-}d_{S^{-1}A}(S^{-1}M).$$

4. Local formulae. For any prime $\mathfrak{p}$ of A, we denote the direct image of a torsion theory $(\mathcal{T}, \mathcal{F})$ under the canonical morphism $A \to A_\mathfrak{p}$, by $(\mathcal{T}_\mathfrak{p}, \mathcal{F}_\mathfrak{p})$. A direct consequence of Proposition 3.2 is

Proposition 4.1. *Let A be a Noetherian ring with a torsion theory $(\mathcal{T}, \mathcal{F})$ and M an A-module; then*

(1) $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \operatorname{Inf}_{\mathfrak{p} \in \operatorname{Spec} A} \{(\mathcal{T}_\mathfrak{p}, \mathcal{F}_\mathfrak{p})\text{-}d_{A_\mathfrak{p}}(M_\mathfrak{p})\}$.

However, it is not necessary to look at all the primes of Spec A.

Corollary 4.2. *Let A be a Noetherian ring with a torsion theory $(\mathcal{T}, \mathcal{F})$ and M an A-module, then*

(2) $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \operatorname{Inf}_{\mathfrak{p} \in \mathbf{T}}\{(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}$ *(where* $(\mathbf{T}, \mathbf{F})$ *is the partition of* Spec A *associated to* $(\mathcal{T}, \mathcal{F})$*).*

(3) $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \operatorname{Inf}_{\mathfrak{p} \in \operatorname{Max} A}\{(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}$ *(where* Max A *is the set of maximal ideals of* A*).*

(4) $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \operatorname{Inf}_{\mathfrak{p} \in \operatorname{Supp}_A(M)}\{(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}$ *(where* $\operatorname{Supp}_A(M)$ *is the support of the A-module M).*

Proof. (2) If $\mathfrak{p} \in \mathbf{F}$ it is clear that $(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})$ is the torsion theory such that every A-module is torsion-free, hence $(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \infty$.

(3) and (4) are equally easy.

The set of minimal primes of $(\mathcal{T}, \mathcal{F})$ is the set of primes which belong to $\mathbf{T}$ and are minimal among the elements of $\mathbf{T}$. We denote this set $\mathbf{T}_0$. As a generalization of Theorem 2.1 we get

Theorem 4.3. *Let A be a Noetherian ring with a torsion theory $(\mathcal{T}, \mathcal{F})$ and let $\mathbf{T}_0$ be the set of minimal primes of $(\mathcal{T}, \mathcal{F})$ and M an A-module of finite type; then*

$$(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \operatorname*{Inf}_{\mathfrak{p} \in \mathbf{T}_0} \{\mathfrak{p}\text{-}\operatorname{Depth}_A(M)\}.$$

Proof. $\mathfrak{p}\text{-}\operatorname{Depth}_A(M)$ is the depth of M relative to the torsion theory whose partition $(\mathbf{T}_{\mathfrak{p}}, \mathbf{F}_{\mathfrak{p}})$ is such that $\mathbf{T}_{\mathfrak{p}} = V(\mathfrak{p})$ [Theorem 2.1], hence this theory is smaller than $(\mathcal{T}, \mathcal{F})$ and:

$$(\mathcal{T}, \mathcal{F})\text{-}\operatorname{Depth}_A(M) \leq \mathfrak{p}\text{-}\operatorname{Depth}_A(M) \quad \forall \mathfrak{p} \in \mathbf{T}_0 \qquad \text{[Proposition 1.6].}$$

The reverse inequality is shown by induction. We just prove the first step, that is, if $\operatorname{Inf}_{\mathfrak{p} \in \mathbf{T}_0}\{\mathfrak{p}\text{-}\operatorname{Depth}_A(M)\} > 0$ then $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) > 0$. Indeed, if this infimum is greater than 0, then M is torsion-free for any of the theories corresponding to the partitions $(\mathbf{T}_{\mathfrak{p}}, \mathbf{F}_{\mathfrak{p}})$, so

$$\operatorname{Ass}_A(M) \cap V(\mathfrak{p}) = \varnothing \quad \forall \mathfrak{p} \in \mathbf{T}_0.$$

Since $\mathbf{T} = \bigcup_{\mathfrak{p} \in \mathbf{T}_0} V(\mathfrak{p})$, $\operatorname{Ass}_A(M) \subset \mathbf{F}$, and M is torsion-free for $(\mathcal{T}, \mathcal{F})$, which is equivalent to $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) > 0$.

From the formula $I\text{-}\operatorname{Depth}_A(M) = \operatorname{Inf}_{\mathfrak{p} \supseteq I}\{\operatorname{Depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}$ [11, 15.6]. We get

Theorem 4.4. *Let A be a Noetherian ring with a torsion theory $(\mathcal{T}, \mathcal{F})$ and M an A-module of finite type; then*

$$(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \operatorname*{Inf}_{\mathfrak{p} \in \mathbf{T}} \{\operatorname{Depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}.$$

B. MODULE OF QUOTIENTS

We let A be a Noetherian ring with a torsion theory $(\mathcal{T}, \mathcal{F})$, and $\mathbf{T}$, $\mathbf{F}$ be the corresponding partition of Spec A.

5. An exact sequence.

Proposition 5.1. *Let M be an A-module and f: $(M) \to Q(M)$ be the canonical morphism of M into its module of quotients, then* Ker f *is functorially isomorphic to the torsion radical $T(M)$ of M, and* Coker f *is functorially isomorphic to $T_1(M)$, where T_1 is the first derived functor of T.*

In other words if g: $M \to N$ is a module homomorphism the diagram below is commutative and its lines are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(M) & \longrightarrow & M & \xrightarrow{f(M)} & Q(M) & \longrightarrow & T_1(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T(g)} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle Q(g)} & & \downarrow{\scriptstyle T_1(g)} & & \\
0 & \longrightarrow & T(N) & \longrightarrow & N & \xrightarrow{f(N)} & Q(N) & \longrightarrow & T_1(N) & \longrightarrow & 0
\end{array}$$

Proof. By definition of $Q(M)$ the kernel of $f(M)$ is the torsion radical $T(M)$ of M, and its cokernel C is a torsion module. If M' denotes the image of M in $Q(M)$, we have two short exact sequences:

(a) $0 \to T(M) \to M \to M' \to 0$,

(b) $0 \to M' \to Q(M) \to C \to 0$

from which we get two long exact sequences:

(c) $\cdots \to T_1(T(M)) \to T_1(M) \to T_1(M') \to T_2(T(M)) \to \cdots$.

(d) $\cdots \to T(Q(M)) \to T(C) \to T_1(M') \to T_1(Q(M)) \to \cdots$.

Since $Q(M)$ is torsion-free and divisible, then $T(Q(M))$ and $T_1(Q(M))$ are both 0 [1.4], and since C is a torsion module $T(C) = C$, so from (d), C is isomorphic to $T_1(M')$. On the other hand $T_1(T(M))$ and $T_2(T(M))$ are also both 0, since $T(M)$ is a torsion module (from the lemma below) and then $T_1(M')$ is isomorphic to $T_1(M)$, which is in turn isomorphic to C. The isomorphisms are functorial since the exact sequences (c) and (d) are functorial.

Lemma 5.2. *If M is a torsion module then $T_n(M) = 0$, $n > 0$.*

Proof. Since M is a torsion module, then $\mathrm{Ass}_A(M) \subset \mathbf{T}$. If M_0 is an injective hull of M, then $\mathrm{Ass}_A(M_0) \subset \mathbf{T}$, since A is Noetherian, hence M_0 is also a torsion module; the quotient M_0/M is a torsion module and, step by step, one can get an injective resolution of M by torsion modules. Applying T to this resolution does not change it, then the homology groups of the resulting complex are all trivial.

6. **Open sets.** When F is an open set U of Spec A, we can give an explicit description of $Q(M)$.

Theorem 6.1. *Let M be an A-module, $\widetilde{M}$ be the quasi-coherent module canonically associated to M over* Spec A, *j the canonical morphism of M into the module of sections $\widetilde{M}(U)$. Then, $j: M \to \widetilde{M}(U)$ can be identified with the canonical morphism of M into its module of quotients.*

Proof. It is enough to show that Ker g and Coker g are torsion modules whereas $\widetilde{M}(U)$ is torsion-free and divisible [9, Proposition 0.7]. By definition of $\widetilde{M}(U)$, if $\mathfrak{p} \in \mathbf{F} = U$ then the localization $j_{\mathfrak{p}}: M_{\mathfrak{p}} \to \widetilde{M}(U)_{\mathfrak{p}}$ is an isomorphism, hence $(\operatorname{Ker} g)_{\mathfrak{p}} = (\operatorname{Coker} g)_{\mathfrak{p}} = 0$ and then $\operatorname{Ass}_A(\operatorname{Ker} g) \subset \operatorname{Supp}_A(\operatorname{Ker} g) \subset \mathbf{T}$; also $\operatorname{Ass}_A(\operatorname{Coker} g) \subset \mathbf{T}$, so the kernel and cokernel of g are torsion modules. U is quasi-compact, since A is Noetherian, and can be covered by finitely many special open sets of Spec A, say $U_1 = D(f_1), \cdots, U_n = D(f_n)$, and $U_i \cap U_j = D(f_i f_j), \forall i, j \in \{1, \cdots, n\}$. Since $\widetilde{M}$ is a sheaf, there is an exact sequence

$$0 \to \widetilde{M}(U) \to \bigoplus_{i=1}^{n} \widetilde{M}(U_i) \to \bigoplus_{i,j} \widetilde{M}(U_i \cap U_j)$$

or also,

$$0 \to \widetilde{M}(U) \to \bigoplus_{i} M_{f_i} \to \bigoplus_{i,j} M_{f_i f_j}.$$

Since $U_i \subset U = \mathbf{F}$, then $\operatorname{Ass}_A(M_{f_i}) \subset U_i \subset \mathbf{F}$ and M_{f_i} is torsion-free, the direct sum $\bigoplus_i M_{f_i}$ is also torsion-free and so is its submodule $\widetilde{M}(U)$. We can write $T(\widetilde{M}(U)) = 0$. For the same reasons $M_{f_i f_j}$ is torsion-free, as well as the direct sum $\bigoplus_{i,j} M_{f_i f_j}$. We denote by P the image of $\bigoplus M_{f_i}$ into $\bigoplus_{i,j} M_{f_i f_j}$, then P is torsion-free. From the short exact sequence $0 \to \widetilde{M}(U) \to \bigoplus_i M_{f_i} \to P \to 0$, we get the long exact sequence $\cdots \to T(P) \to T_1(\widetilde{M}(U)) \to T_1(\bigoplus_i M_{f_i}) \to \cdots$ (*) where $T(P) = 0$. Since $U_i \subset \mathbf{F}$, the torsion theory $(\mathcal{T}_i, \mathcal{F}_i)$, direct image of $(\mathcal{T}, \mathcal{F})$ over A_{f_i} is clearly trivial [§ 3], every A_{f_i}-module is torsion-free for this theory, in other words, one check easily that

$$(\mathcal{T}_i, \mathcal{F}_i)\text{-}d_{A_{f_i}}(N) - \infty, \quad \text{for every } A_{f_i}\text{-module } N.$$

Then $(\mathcal{T}, \mathcal{F})\text{-}d_A(N) = \infty$ [Proposition 3.2], and in particular $T(M_{f_i}) = T_1(M_{f_i}) = 0$, and easily $T_1(\bigoplus_i M_{f_i}) = 0$. From (*), it results that $T_1(\widetilde{M}(U)) = 0$, and since also $T(\widetilde{M}(U)) = 0$, $\widetilde{M}(U))$ is torsion-free and divisible.

7. **Localization.** We show here that over a Noetherian ring, the functor Q commutes with localization with respect to a multiplicative subset of A. The result is easy but useful for the following.

Proposition 7.1. *Let S be a multiplicative subset of A, M an A-module and $f: M \to Q(M)$ be the canonical morphism of M into its module of quotients. Then $S^{-1}f: S^{-1}M \to S^{-1}Q(M)$ is the canonical morphism of the $S^{-1}A$-module $S^{-1}M$ into its ring of quotients for the theory $(S^{-1}\mathfrak{T}, S^{-1}\mathfrak{F})$, direct image of $(\mathfrak{T}, \mathfrak{F})$ under the morphism $A \to S^{-1}A$.*

Proof. Since $Q = Q(M)$ is torsion-free and divisible then $(\mathfrak{T}, \mathfrak{F})\text{-}d_A(Q) \geq 2$, but then $(S^{-1}\mathfrak{T}, S^{-1}\mathfrak{F})\text{-}d_{S^{-1}A}(S^{-1}Q) \geq 2$ [Proposition 3.2] and $S^{-1}Q$ is a torsion-free and divisible $S^{-1}A$-module. The kernel and cokernel of f are torsion A-modules, their localization are torsion A-modules and then torsion $S^{-1}A$-modules, from the following lemma:

Lemma 7.2. *Let A be any commutative ring, S a multiplicative subset of A, and M a torsion A-module, then $S^{-1}M$ is a torsion A-module.*

Proof. Every element of $S^{-1}M$ can be written x/s, where $x \in M$, and $s \in S$, and the submodule Ax/s of $S^{-1}M$ is an homomorphic image of the submodule Ax of M.

It is interesting to note that such a property does not always hold for torsion-free modules [3, § 1, Corollaire 2].

C. AFFINE TORSION THEORY

8. **(T)-condition of Goldman.** In general $Q(M)$ is not isomorphic to $M \otimes_A Q(A)$. If this holds for every module M, Goldman says that $(\mathfrak{T}, \mathfrak{F})$ satisfied the (T) condition (T for tensor). We quote [5, Theorem 4.3]:

Theorem 8.0. *The following conditions are equivalent:*

(i) (T) *condition*: $Q(M) \cong M \otimes_A Q(A)$; *for every A-module M,*

(ii) *every $Q(A)$-module is torsion-free,*

(iii) $\mathfrak{b}\, Q(A) = Q(A)$, *for every ideal $\mathfrak{b}$ of A such that $A/\mathfrak{b}$ is a torsion module,*

(iv) *Q is exact and commutes with direct sums.*

We want to show that the (T) condition is related to the set of free-primes F being an affine subset of Spec A. If however F is not an open subset of Spec A, the notion affine subset would make sense only in the theory of "Espaces Étales" [4, II, § 1–2] but Lazard proved that F is affine if and only if there is a flat epimorphism of rings $f: A \to B$, such that F is the set of primes of A which are lifted in B [10, IV, Proposition 2.5]. We can take this characterization as a definition, for open sets it gives back the usual notion of affine open sets.

We denote by f the canonical morphism of A into its ring of quotients $Q(A)$, and by ${}^af$ the morphism ${}^af$: Spec $Q(A) \to$ Spec A canonically associated to f.

Lemma 8.1. *For every prime* $\mathfrak{p}$ *of* **F**, *the localization* $f_{\mathfrak{p}}: A_{\mathfrak{p}} \to Q(A)_{\mathfrak{p}}$ *of* f *is an isomorphism.*

Proof. Let K and C be the kernel and cokernel of f. Then $K_{\mathfrak{p}}$ and $C_{\mathfrak{p}}$ are the kernel and cokernel of $f_{\mathfrak{p}}$. Their associated primes are clearly included in $\mathfrak{p}$, then belong to **F**, since $\mathfrak{p} \in \mathbf{F}$ and **F** is closed under generization: $\mathrm{Ass}_A(K_{\mathfrak{p}}) \subset \mathbf{F}$, and $\mathrm{Ass}_A(C_{\mathfrak{p}}) \subset \mathbf{F}$ so $K_{\mathfrak{p}}$ and $C_{\mathfrak{p}}$ are torsion-free modules [§ 0]. But K and C are torsion modules, hence $K_{\mathfrak{p}}$ and $C_{\mathfrak{p}}$ are also torsion modules [Lemma 7.2]. Being both torsion and torsion-free they are trivial.

From this lemma it is clear that $\mathbf{F} \subset {}^a f[\mathrm{Spec}\ Q(A)]$.

Theorem 8.2. *The followings are equivalent:*

(i) $(\mathcal{T}, \mathcal{F})$ *satisfies the* (T)*-condition of Goldman.*

(ii) $\mathbf{F} = {}^a f[\mathrm{Spec}\ Q(A)]$.

(iii) **F** *is affine and* $(\mathcal{T}, \mathcal{F})$ *is well-centered.*

Proof. (i) $\Longrightarrow$ (ii). If $(\mathcal{T}, \mathcal{F})$ satisfies the (T)-condition of Goldman, and if $\mathfrak{p} \in \mathbf{T}$, then $A/\mathfrak{p}$ is torsion [§ 0], and $\mathfrak{p}Q(A) = Q(A)$ [Theorem 8.0(iii)]. Hence there are no primes in $Q(A)$ lifting $\mathfrak{p}$, ${}^a f[\mathrm{Spec}\ Q(A)]$ is included in **F**, in fact is equal to **F**, since the reverse inclusion is true [Lemma 8.1].

(ii) $\Longrightarrow$ (iii). If $\mathbf{F} = {}^a f[\mathrm{Spec}\ Q(A)]$; then $\forall \mathfrak{q} \in \mathrm{Spec}\ Q(A)$, $f^{-1}(\mathfrak{q}) \in \mathbf{F}$ and $f \otimes_A A_{f^{-1}(\mathfrak{p})}$ is an isomorphism [Lemma 8.1], hence f is a flat epimorphism [10, IV, Proposition 2.4]. Since f is a flat epimorphism and **F** is the set of primes which are lifted in $Q(A)$, **F** is affine. Now, if M is torsion-free, M is a submodule of $Q(M)$ and $Q(M)$ is a $Q(A)$-module, then

$$\mathrm{Ass}_A(M) \subset \mathrm{Ass}_A(Q(M)) \subset {}^a f[\mathrm{Ass}_{Q(A)}(Q(M))] \subset {}^a f[\mathrm{Spec}\ Q(A)] = \mathbf{F}$$

(for the second inclusion [10, II, Proposition 3.1]). Then $(\mathcal{T}, \mathcal{F})$ is well cen- [§ 0].

(iii) $\Longrightarrow$ (i). If **F** is affine there is a flat epimorphism $g: A \to B$ such that $\mathbf{F} = {}^a g[\mathrm{Spec}\ B]$. There is also a torsion theory $(\mathcal{T}', \mathcal{F}')$ such that g is the canonical morphism of A into its ring of quotients $Q'(A) = B$ [9, Proposition 2.7], and such that every B-module is torsion-free. $(\mathcal{T}', \mathcal{F}')$ satisfies the (T)-condition of Goldman [Theorem 8.0(ii)], and then the set of free-primes of $(\mathcal{T}', \mathcal{F}')$ is ${}^a g[\mathrm{Spec}\ B] = \mathbf{F}$, since (i) $\Longrightarrow$ (ii). $(\mathcal{T}', \mathcal{F}')$ is well-centered since (ii) $\Longrightarrow$ (iii), then, in fact, $(\mathcal{T}', \mathcal{F}')$ is the same torsion theory as $(\mathcal{T}, \mathcal{F})$ since they both have the same set of free-primes **F**, and are both well-centered, hence both have the same torsion-free modules [§ 0].

Definition 8.3. If $(\mathcal{T}, \mathcal{F})$ satisfies the (T)-condition of Goldman, we say also that $(\mathcal{T}, \mathcal{F})$ if affine.

Remarks. If Q is exact, $(\mathcal{T}, \mathcal{F})$ is not necessarily affine [1, II, § 2, Exercise 20]. **F** may be affine, but $(\mathcal{T}, \mathcal{F})$ not well-centered [(2)]. f may be a flat epimorphism, but $(\mathfrak{T}, \mathfrak{F})$ not affine, even if the ring A is Noetherian: take A the ring of polynomials in two indeterminates X and Y over any field K. The torsion theory such that the maximal ideal (X, Y) is the only torsion prime of A, is such that $Q(A) = A$, however the complement of (X, Y) in Spec A is not affine.

Proposition 8.4. *If* $(\mathfrak{T}, \mathfrak{F})$ *is affine the dominant dimension* $(\mathfrak{T}, \mathfrak{F})\text{-}d_A(M)$ *of any A-module* M *can take only the values* 0, 1 *or* ∞.

Proof. If $(\mathfrak{T}, \mathfrak{F})$ is affine, the morphism $f: A \to Q(A)$ is a flat epimorphism. We denote the direct image of $(\mathfrak{T}, \mathfrak{F})$ under f, by $(\mathfrak{T}_f, \mathfrak{F}_f)$; since every $Q(A)$-module is torsion-free, the $(\mathfrak{T}_f, \mathfrak{F}_f)$-dominant dimension of every such module is infinite. If N is an A-module such that $(\mathfrak{T}, \mathfrak{F})\text{-}d_A(N) \geq 2$, then N is torsion-free and divisible, $N = Q(N)$ is a $Q(A)$-module and $(\mathfrak{T}_f, \mathfrak{F}_f)\text{-}d_{Q(A)}(N) = \infty$, but $(\mathfrak{T}, \mathfrak{F})\text{-}d_A(N) = (\mathfrak{T}_f, \mathfrak{F}_f)\text{-}d_{Q(A)}(N)$, since f is flat [Proposition 3.1].

Remark. Morita has shown independently that $(\mathfrak{T}, \mathfrak{F})$-dominant dimension can take only the values 0, 1 and ∞ if and only if Q is exact [12].

9. **Noetherian rings.** We suppose now that A is Noetherian. For any prime $\mathfrak{p}$ of A, we let $(\mathfrak{T}_\mathfrak{p}, \mathfrak{F}_\mathfrak{p})$ be the direct image of the torsion theory $(\mathfrak{T}, \mathfrak{F})$, under the localization $A \to A_\mathfrak{p}$ [§3], and $Q_\mathfrak{p}$ the quotient functor in the category Mod A of $A_\mathfrak{p}$-modules, relative to this theory.

Proposition 9.1. *If* A *is Noetherian the following statements are equivalent.*

(i) $(\mathfrak{T}, \mathfrak{F})$ *is affine.*

(ii) *For every prime* $\mathfrak{p}$ *of* Spec A, $(\mathfrak{T}_\mathfrak{p}, \mathfrak{F}_\mathfrak{p})$ *is affine.*

Proof. The localization $f_\mathfrak{p}$, can be identified with the canonical morphism of $A_\mathfrak{p}$ into its ring of quotients $Q_\mathfrak{p}(A_\mathfrak{p})$ for $(\mathfrak{T}_\mathfrak{p}, \mathfrak{F}_\mathfrak{p})$ [Proposition 7.1] also the set of free-primes for $(\mathfrak{T}_\mathfrak{p}, \mathfrak{F}_\mathfrak{p})$ in $A_\mathfrak{p}$ is the set of primes $\mathbf{F}_\mathfrak{p}$, corresponding to the primes of $\mathbf{F}$ contained in $\mathfrak{p}$ [§3], hence $\mathbf{F} = {}^af[\text{Spec } Q(A)]$ if and only if $\mathbf{F}_\mathfrak{p} = {}^af_\mathfrak{p}[\text{Spec } Q_\mathfrak{p}(A_\mathfrak{p})]$ for every $\mathfrak{p}$.

Theorem 8.2 becomes also much simpler:

Theorem 9.2. *If* A *is Noetherian, the following statements are equivalent:*

(i) $(\mathfrak{T}, \mathfrak{F})$ *is affine.*

(ii) Q *is exact.*

(iii) *For every module* M, $T_2(M) = 0$.

(iv) *For every module* M, $T_n(M) = 0 \ \forall\, n \geq 2$.

(v) *For every module* M, $(\mathfrak{T}, \mathfrak{F})\text{-}d_A(M) = 0, 1$ *or* ∞.

Proof. (i) $\Rightarrow$ (ii). [5, Theorem 4.4].

(ii) $\Rightarrow$ (iii). For every module M there is a functorial exact sequence $0 \to T(M) \to (M) \to Q(M) \to T_1(M) \to 0$ [Proposition 5.1]. Then if $\phi: M \to P$ is an onto map, there is a commutative square

$$(*)\qquad \begin{array}{ccccc} \to Q(M) & \to & T_1(M) & \to & 0 \\ \downarrow Q(\phi) & & \downarrow T_1(\phi) & & \\ \to Q(P) & \to & T_1(P) & \to & 0 \end{array}$$

where $Q(\phi)$ is onto. Then T_1 is right exact and it follows easily that T_2 is trivial.

(iii) $\Rightarrow$ (iv). Easy by induction on n, from $n = 2$.

(iv) $\Rightarrow$ (v). Obvious.

(v) $\Rightarrow$ (iv). If M is torsion-free and divisible, then $T(M) = T_1(M) = 0$ and thus $T_n(M) = 0$ $\forall n \geq 0$. If M is a torsion module $T_n(M) = 0$ $\forall n \geq 1$ [Lemma 5.2]. Consequently if M is torsion-free, using the exact sequence $0 \to M \to Q(M) \to T_1(M) \to 0$ where $Q(M)$ is torsion-free and divisible and where $T_1(M)$ is torsion, one can prove that $T_n(M) = 0$ $\forall n \geq 2$. Finally if M is any A-module, using the exact sequence $0 \to T(M) \to M \to M/T(M) \to 0$, where $T(M)$ is torsion and where $M/T(M)$ is torsion-free, one can prove that $T_n(M) = 0$ $\forall n \geq 2$.

(iv) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (ii). If T_1 is right exact, then Q also is right exact, using again the commutative square (*). Since Q is always left exact, Q is exact.

10. Applications: affine open sets. If A is Noetherian and if U is an open set of Spec A, there is a unique torsion theory over A such that $U = \mathbf{F}$. Also if we denote Spec A by X, and the quasi-coherent $\mathcal{O}_X$-module canonically associated to an A-module M by $\widetilde{M}$ then $\widetilde{M}(U)$ is functorially isomorphic to $Q(M)$ [Theorem 6.1]. The following proposition, a direct consequence of Theorem 9.2, is similar to Serre's theorem [15, Theorem 1].

Proposition 10.1. *Let A be a Noetherian ring and U an open set of* Spec A, *the following statements are equivalent*:

(i) *U is affine.*

(ii) *For every module M, $\widetilde{M}(U) \cong M \otimes_A \mathcal{O}_X(U)$.*

(iii) *The functor $M \to \widetilde{M}(U)$ is exact in* Mod A.

We now characterize the affine open sets of a regular Noetherian ring, that is to say a Noetherian ring A such that $A_{\mathfrak{p}}$ is a regular local ring at every prime $\mathfrak{p}$. If I is an ideal of A, we let $V(I)$ be the set of primes containing I and $h(I)$, the height of I, be the maximum of the heights of the minimal primes of I. First we suppose only that A is Cohen-Macaulay, that is to say that Depth $(A_{\mathfrak{p}}) = h(\mathfrak{p})$, for every prime $\mathfrak{p}$.

Proposition 10.2. *Let A be a Noetherian Cohen-Macaulay ring and $(\mathfrak{T}, \mathfrak{F})$ an affine torsion theory over A. If $\mathfrak{p} \in T_0$, that is $\mathfrak{p}$ is minimal among the torsion-primes, then $h(\mathfrak{p}) \leq 1$.*

Proof. Since $\mathfrak{p} \in T_0$, $\mathfrak{p}A_{\mathfrak{p}}$ is the only torsion-prime of the theory $(\mathfrak{T}_{\mathfrak{p}}, \mathfrak{F}_{\mathfrak{p}})$, direct image of $(\mathfrak{T}, \mathfrak{F})$ under the localization $A \to A_{\mathfrak{p}}$, hence $(\mathfrak{T}_{\mathfrak{p}}, \mathfrak{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(A\mathfrak{p}) =$ Depth $(A_{\mathfrak{p}})$ [Theorem 2.1]. But Depth $(A_{\mathfrak{p}}) = h(\mathfrak{p}) < \infty$, and since $(\mathfrak{T}, \mathfrak{F})$ is affine, $(\mathfrak{T}_{\mathfrak{p}}, \mathfrak{F}_{\mathfrak{p}})$ also is affine [Proposition 9.1] and $(\mathfrak{T}_{\mathfrak{p}}, \mathfrak{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = h(\mathfrak{p}) \leq 1$ [Proposition 8.4].

Theorem 10.3. *Let A be a regular Noetherian ring, I an ideal of A, and U the complement of $V(I)$ in* Spec *A, then*

(1) *U is an affine open set if and only if $ht(I) \leq 1$.*

(2) *If A is local and U is affine, then U is a special open set, that is U is the set of primes which do not contain an element f of A.*

Proof. A regular Noetherian ring is also Cohen-Macaulay. If U is affine, since the minimal primes of I are also the minimal torsion-primes of the theory such that $F = U$, then from the previous proposition $h(I) \leq 1$. Conversely if $h(I) \leq 1$, then, for every prime $\mathfrak{p}$, $h(I_{\mathfrak{p}}) \leq 1$; also to prove that U is affine it is enough to prove that for every $\mathfrak{p}$, the complement $U_{\mathfrak{p}}$ of $V(I_{\mathfrak{p}})$ in Spec $A_{\mathfrak{p}}$ is affine [Proposition 9.1]. Then we can assume that A is local, so it is also a unique factorization domain. If $ht(I) = 0$, then $U =$ Spec A is clearly a special open set. If $ht(I) = 1$, the minimal primes of I must all be of height 1, and thus be principal. If $p_1, p_2 \cdots p_n$ are respective generators of those primes it is easy to check that U is the set of primes which do not contain $f = p_1 \cdot p_2 \cdots p_n$.

Remark. This theorem is similar to the characterization of affine open sets, complement of a nonsingular variety in the projective space [7, II, Proposition 3.1].

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